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On Geometries of the Fischer Groups

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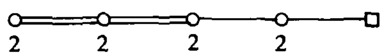
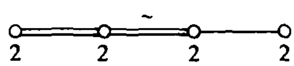
The Fischer groups Fi_{22}, Fi_{23} and Fi_{24} are associated with a number of diagram geometries, some of which have been characterized recently. Exploiting a relationship between these geometries we obtain some further characterizations.

1. Introduction

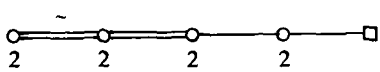
The maximal and minimal 2-local parabolic geometries of the Fischer group F_{24} belong respectively to the following diagrams:

$\mathcal{P}_{24}$: 

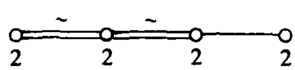
and

$\mathcal{Q}_{24}$: 

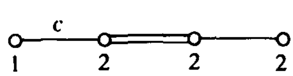
while their analogous associated with the non-split triple cover $3 \cdot Fi_{24}$ belong respectively to the diagrams

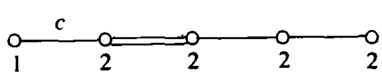
$3.\mathcal{P}_{24}$: 

and

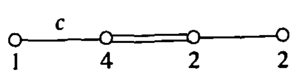
$3.\mathcal{Q}_{24}$: 

The Fischer groups Fi_{22}, Fi_{23} and Fi_{24} act flag-transitively on c-extended dual polar spaces over $GF(2)$ with the diagrams

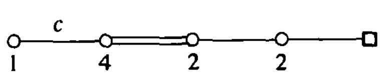
$\mathcal{H}_{22}$: 

$\mathcal{H}_{23}$: 

and

$\mathcal{H}_{24}$: 

respectively. The geometry $\mathcal{H}_{24}$ comes from a locally truncated extension with the diagram

$\mathcal{H}_{24}$: 

The geometry $\mathcal{H}_{22}$ possesses a triple cover $3.\mathcal{H}_{22}$ acted on flag-transitively by a

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non-split extension $3 \cdot Fi_{22}$. The residual $c.C_2$-geometries in $\mathcal{H}_{22}$, $3.\mathcal{H}_{22}$ and $\mathcal{H}_{23}$ are isomorphic to the one associated with the group $U_4(2)$.

The main results of the paper are contained in the following two theorems.

THEOREM A. *The geometries $\mathcal{P}_{24}$, $\mathcal{Q}_{24}$, $3.\mathcal{P}_{24}$, $3.\mathcal{Q}_{24}$, $\mathcal{H}_{24}$ and $\mathcal{H}_{24}$ are simply connected.*

THEOREM B. *Let $\mathcal{H}$ be a flag-transitive c-extension of a dual polar space of symplectic type over $GF(2)$ with a residue isomorphic to the $c.C_2$-geometry of $U_4(2)$. Then $\mathcal{H}$ is isomorphic to $\mathcal{H}_{22}$, $3.\mathcal{H}_{22}$ or $\mathcal{H}_{23}$.*

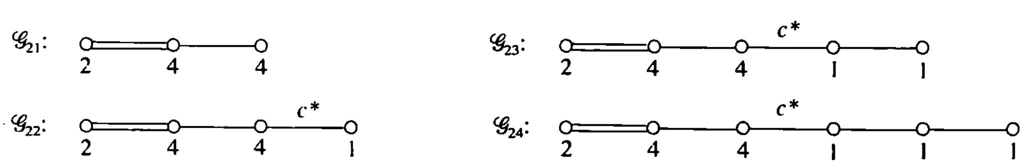
2. THE FISCHER GROUPS

The first family of diagram geometries associated with the Fischer groups Fi_{22}, Fi_{23} and Fi_{24} that we will consider consists of (multiple) extensions of a rank 3 polar space of unitary type. These extensions come from the original construction by B. Fischer of his groups in [7] (see also [4]) and recently have been characterized.

Recall that the Fischer groups Fi_{21} ($\simeq U_6(2)$), Fi_{22}, Fi_{23} and Fi_{24} are groups generated by 3-transpositions. This means that Fi_{21+s}, $0 \leq s \leq 3$ is generated by a conjugacy class D_{21+s} of its involutions (elements of order 2) such that the product of any two different involutions from D_{21+s} has order 2 or 3. Let Γ_{21+s} be a graph on the set D_{21+s} in which two involutions are adjacent exactly when they commute (i.e. when their product is of order 2). Then for $1 \leq s \leq 3$ the graph Γ_{21+s} is locally Γ_{21+s-1}. By the definition this means that the subgraph in Γ_{21+s} induced by the neighbours of a vertex is isomorphic to Γ_{21+s-1}. Moreover, the subgroup in F_{21+s} generated by the neighbours (considered as involutions from D_{21+s}), modulo the subgroup (of order 2) generated by the vertex itself, is isomorphic to F_{21+s-1}. The group $Fi_{21} \simeq U_6(2)$ is classical and D_{21} is the class of unitary transvections, naturally identified with the set of points of the polar space (we denote this polar space by $\mathcal{G}_{21}$) associated with Fi_{21}. The polar space itself possesses a natural description in terms of the graph Γ_{21}. Really, a maximal clique in Γ_{21} consists of 21 points incident to a plane in the polar space. The intersection of two different maximal cliques is either empty or a single point or contains 5 points incident to a line. This means that the elements of $\mathcal{G}_{21}$ can be identified with the set of intersections of (not necessarily different) maximal cliques. The type is determined by the cardinality and the incidence relation is defined by the inclusion.

Because of the locality property, the structure of maximal cliques in Γ_{21+s} for $1 \leq s \leq 3$ is determined by that in Γ_{21+s-1}. In particular, the maximal cliques in Γ_{21+s} have size $21+s$ and the possible sizes of their intersections are $21+s$, $5+s$, $1+s$, $s, \ldots, 1, 0$. It can be shown that Γ_{21+s-1} always contains a pair of disjoint maximal cliques and this implies that all the above possibilities are realized.

We can associate with Fi_{21+s} a geometry $\mathcal{G}_{21+s}$ which is an extension of the polar space $\mathcal{G}_{21}$. The elements of $\mathcal{G}_{21+s}$ are the subsets of vertices of Γ_{21+s} which are intersections of maximal cliques. We define the types of elements to be $1, 2, \ldots, s+3$ and to correspond to the decreasing sequence of sizes of elements. The incidence is defined by the inclusion. Then $\mathcal{G}_{21+s}$ is an s times extended polar space $\mathcal{G}_{21}$ and we have geometries with the following diagrams (the types in the diagrams increase from left to right):



Clearly, Fi_{21+s} acts flag-transitively on $\mathcal{G}_{21+s}$. The full automorphism group of Fi_{22} contains Fi_{22} as a subgroup of index 2 and it acts flag-transitively on $\mathcal{G}_{22}$. The derived group Fi'_{24} of Fi_{24} has index 2 in the latter and acts flag-transitively on $\mathcal{G}_{24}$. Besides that, there are no other flag-transitive actions on the geometries $\mathcal{G}_{21+s}$ for $1 \leqslant s \leqslant 3$.

The group Fi'_{24} possesses a non-split central extension $3 \cdot Fi'_{24}$ which gives rise a covering $3.\Gamma_{24} \rightarrow \Gamma_{24}$ [15]. This covering is an isomorphism when restricted to the neighbourhood of a vertex. So $3.\Gamma_{24}$ is also locally Γ_{23} and by the same rule we can associate with this graph a geometry $3.\mathcal{G}_{24}$ which is a triple cover of $\mathcal{G}_{24}$. In particular, $\mathcal{G}_{24}$ and $3.\mathcal{G}_{24}$ correspond to the same diagram. The full automorphism groups of $3.\Gamma_{24}$ and $3.\mathcal{G}_{24}$ are both isomorphic to a non-central extension $3 \cdot Fi_{24}$.

PROPOSITION 2.1 [19]. *The geometries $\mathcal{G}_{22}$ and $\mathcal{G}_{23}$ are simply connected and $3.\mathcal{G}_{24}$ is the universal cover of $\mathcal{G}_{24}$.*

The following result has been proved in [14] (see also [1]) and a part of it has been already obtained in [5].

PROPOSITION 2.2. *Let $\mathcal{G}$ be a flag-transitive geometry the diagram of which is that of $\mathcal{G}_{21+s}$ for $1 \leqslant s \leqslant 3$. Suppose that the C_3-residue in $\mathcal{G}$ is isomorphic to the polar space of the group $Fi_{21} \simeq U_6(2)$. Then $\mathcal{G}$ is isomorphic either to $\mathcal{G}_{21+s}$ or to $3.\mathcal{G}_{24}$.*

Substantial progress has been made in classification of the flag-transitive C_3-geometries (cf. Propositions 1.9 in [18] and 3.1 in [26]). The following is a consequence of these results.

PROPOSITION 2.3. *Let $\mathcal{G}$ be a flag-transitive C_3-geometry the diagram of which (including the indices) is that of $\mathcal{G}_{21}$. Then $\mathcal{G} \simeq \mathcal{G}_{21}$.*

COROLLARY 2.4. *Let $\mathcal{G}$ be a flag-transitive geometry the diagram of which (including the indices) is that of $\mathcal{G}_{21+s}$ for $0 \leqslant s \leqslant 3$. Then $\mathcal{G}$ is isomorphic to $\mathcal{G}_{21+s}$ or to $3.\mathcal{G}_{24}$.*

It worth mentioning that in [16] and [17] the extended polar spaces of the Fischer groups were characterized by their diagrams and rank 2 residues without any assumptions on the automorphism group.

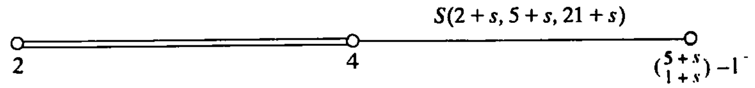
In what follows, the type set of a rank n geometry is assumed to be $\{1, 2, \ldots, n\}$ and in diagrams the type always increases from left to right. The collinearity graph of a geometry is a graph on the set of elements of type 1 where two elements are adjacent if they are incident to a common element of type 2. All geometries in the paper are residually connected either by their construction or by definition.

3. RECONSTRUCTION FROM TRUNCATIONS

In this section we consider truncations $\mathcal{G}^T_{21+s}$ of the geometries $\mathcal{G}_{21+s}$ by the elements of type greater than 3. These truncations play an intermediate role between the geometries $\mathcal{G}_{21+s}$ and other geometries of the Fischer groups.

Consider an element of type 1 in $\mathcal{G}_{21+s}$; that is, a maximal clique K in Γ_{21+s}. The elements of type 2 incident to K, and considered as $(5+s)$-element subsets of K, define on K a structure of the Steiner system $S(2+s, 5+s, 21+s)$, which is the unique s times extended projective plane of order 4, admitting the Mathieu group M_{21+s} as an automorphism group. Here, as usual, $M_{21} \simeq L_3(4)$. Every $(1+s)$-element subset in K is an element of type 3. This means that the truncation $\mathscr{G}^T_{21+s}$ of $\mathscr{G}_{21+s}$ by the elements with type greater than 3 corresponds to the following diagram:

$\mathscr{G}^T_{21+s}$:



where the right edge denotes the rank 2 geometry of blocks and all $(1+s)$-element subsets of points in the Steiner system $S(2+s, 5+s, 21+s)$ with the natural incidence relation. Of course, for $s=0$ we have just the projective plane of order 4. A similar truncation of $3.\mathscr{G}_{24}$ will be denoted by $3.\mathscr{G}^T_{24}$.

To be able to reconstruct $\mathscr{G}_{21+s}$ from $\mathscr{G}^T_{21+s}$ we will make use of certain properties of automorphism groups. Let $G \simeq Fi_{21+s}$ act on $\mathscr{G}^T_{21+s}$ for some $1 \leq s \leq 3$ or $G \simeq Fi_{22}.2$ act on $\mathscr{G}^T_{22}$ and, for an element x in the geometry, let $G(x)$ denote the stabilizer of x in G. Let x and y be incident elements of type 1 and 2, respectively and let $x_1 = x$, x_2 and x_3 be all the elements of type 1 incident to y. The properties listed below are well known. In order to check them it is convenient to think of the elements in the geometry as of the corresponding cliques in the transposition graph.

(i) $G(x) \simeq 2^{9+s}.M_{21+s}$ or $2^{10}.M_{22}.2$, where $G(x)/O_2(G(x))$ is the group induced by $G(x)$ on the Steiner system $S(x)$ which is the residue of x. Let $V(x)$ be the $GF(2)$-space of all the subsets of points in $S(x)$. Then $O_2(G(x))$ is a module for M_{21+s} or $M_{22}.2$ isomorphic to an irreducible factor of $V(x)$ over a submodule which contains all blocks of $S(x)$. This means that $O_2(G(x))$ is isomorphic to the unique $(9+s)$-dimensional section of the Golay cocode considered as a module for M_{21+s}.

(ii) $G(x) \cap G(y) \simeq 2^{9+s}.2^4.A_{5+s}$ or $2^{10}.2^4.S_6$ is the full preimage in $G(x)$ of the stabilizer in $G(x)/O_2(G(x))$ of a block B from $S(x)$. For $i = 2$ or 3, $O_2(G(x)) \cap G(x_i)$ is the image in $O_2(G(x))$ of the hyperplane in $V(x)$ consisting of the subsets with even intersections with the complement of B and $O_2(G(x)) \cap O_2(G(x_i))$ is the image of the submodule in $V(x)$ generated by the elements of B. As a module for A_{5+s} or S_6, the latter is the natural permutation module factorized over the submodule of constant functions.

(iii) $G(y) \simeq 2^{4+s}.2^8.(A_{5+s} \times S_3)$ or $2^5.2^8.(S_6 \times S_3)$, where $O_2(G(y))$ is a special group generated by $O_2(G(x_i))$'s for $i = 1, 2$ and 3; its centre $Z(y)$ coincides with $O_2(G(x_i)) \cap O_2(G(x_j))$ for $1 \leq i \neq j \leq 3$ and the S_3-factor of $G(y)/O_2(G(y))$ acts trivially on $Z(y)$.

Proposition 3.1. *Let $\mathscr{G}$ be a rank 3 geometry the diagram of which is that of $\mathscr{G}^T_{21+s}$ for some $1 \leq s \leq 3$. Let G be a flag-transitive automorphism group of $\mathscr{G}$. Suppose that the stabilizer $G(x)$ of an element of type 1 induces on the residue of x the natural action of M_{21+s} or $M_{22}.2$ and the kernel of this action induces on the set of elements of type 1 adjacent to x in the collinearity graph an elementary abelian group of order 2^{9+s} isomorphic as a module for M_{21+s} to an irreducible section of the Golay cocode. Then $\mathscr{G}$ is isomorphic to $\mathscr{G}^T_{21+s}$ or to $3.\mathscr{G}^T_{24}$.*

Proof. Let Ξ be the collinearity graph of $\mathscr{G}$ and let $\Xi(x)$ be the set of vertices adjacent to x. By the hypothesis $G(x)$ induces on $\Xi(x)$ an action isomorphic to $2^{9+s}.M_{21+s}$ or $2^{9+s}.M_{22}.2$. We claim that the action is faithful. Let K be the kernel of the action. Clearly, all we have to show is that K acts trivially on $\Xi(u)$ for some $u \in \Xi(x)$. Let z be an element of type 3 incident to both x and u. It follows from the well known properties of the automorphism group of the residue of z that K must act trivially on this residue. Now, the available information on the action of $G(u)$ on $\Xi(u)$ is sufficient to prove that the element-wise stabilizer in $G(u)$ of the elements from $\Xi(u)$, which are contained in the residue of an element of type 3 incident to x and u, acts trivially on $\Xi(u)$.

We will identify the elements of $\mathcal{G}$ with certain subgraphs in Ξ. As above, let x be a vertex of Ξ (an element of type 1). Let $S(x)$ be the corresponding Steiner system with the point set $P(x)$ and with the block set $B(x)$. By the above paragraph, $O_2(G(x))$ is generated by a set of involutions indexed by (and identified with) points in $P(x)$. Let $B \in B(x)$ and let y be the element of type 2 incident to x which corresponds to B. Let $x_1 = x$, x_2, x_3 be the elements of type 1 incident to y. Let $Z(y)$ be the subgroup in $O_2(G(x))$ generated by the points in B. Then one can show that $Z(y)$ is the centre of $O_2(G(x_1) \cap G(x_2) \cap G(x_3))$. Moreover, x_2 and x_3 are the only points adjacent to x in Ξ which are fixed by $Z(y)$ and hence the normalizer of $Z(y)$ in $G(y)$ induces on $\{x_1, x_2, x_3\}$ the symmetric group S_3. This enables us to identify y with the triangle $\Xi_2 = \{x_1, x_2, x_3\}$, which is the connected component containing x of the subgraph induced by the vertices of Ξ fixed by $Z(y)$. Let A be a $(1+s)$-element subset in $P(x)$ and let z be the corresponding element of type 3 incident to x. Then z can be identified with a 27-vertex subgraph Ξ_3 in Ξ which is the connected component containing x of the subgraph induced by the vertices fixed by the subgroup of $O_2(G(x))$ generated by the points in A. Now, all the elements of $\mathcal{G}$ are the images under G of the subgraphs $\Xi_1 = \{x\}$, Ξ_2 and Ξ_3, with the incidence relation defined by the inclusion. The elements of the missed types can be defined in a similar way: namely, for every $1 \leqslant t \leqslant s$, let Ξ_{3+t} be the connected component containing x of the subgraph in Ξ induced by the vertices fixed by the subgroup in $O_2(G(x))$ generated by the points in a $(1+s-t)$-element subset of $P(x)$. We define the elements of type $3+t$ to be the images under G of the subgraph Ξ_{3+t}. The incidence relation between both old and new elements is defined by the inclusion. The diagram of $\mathcal{G}$ implies almost immediately that the diagram of the resulting (rank $3+s$) geometry coincides with that of $\mathcal{G}_{21+s}$ and the claim follows by Corollary 2.3. □

By Proposition 3.1, the geometries $\mathcal{G}_{22}^T$, $\mathcal{G}_{23}^T$ and $3.\mathcal{G}_{24}^T$ are simply connected, and hence we have the following.

COROLLARY 3.2. *Let G be a flag-transitive automorphism group of $\mathcal{G}_{21+s}^T$ for $s = 1$, 2 or 3 (that is, $G \simeq Fi_{22}$, $Fi_{22}.2$, Fi_{23}, Fi_{24} or Fi_{24}'). Let G_i be the stabilizer in G of the element of type i from a fixed maximal flag in the geometry and let $\mathcal{A}$ be the amalgam consisting of these stabilizers. Then $\mathcal{G}$ is the unique completion of $\mathcal{A}$ for $s = 1$ or 2; for $s = 3$ there is one additional completion, namely $3 \cdot G$.*

In what follows, G_i denotes the stabilizer in $G \simeq Fi_{21+s}$ of the element of type i in the fixed maximal flag of $\mathcal{G}_{21+s}$, where s will be clear from the context. Notice that in all cases $G_i = N_G(O_2(G_i))$.

4. PARABOLIC GEOMETRIES

Let $\mathcal{P}_{22}$ be the minimal parabolic geometry of the group Fi_{22} from [22] for which we adopt the reverse ordering of types. Let P_i be the stabilizer in $G \simeq Fi_{22}$ of the element of type i in a maximal flag in $\mathcal{P}_{22}$, $i = 1$, 2 and 3. Then, for a suitable choice of the flag, $P_1 = G_1$, $P_3 = G_3$ and P_2 is a proper subgroup of G_2, which can be specified in the following way. Let x be the element of type 1 in $\mathcal{G}_{22}^T$ stabilized by G_1, let

$S(x) = (P(x), B(x))$ be the corresponding Steiner system, and let $B \in B(x)$ be the block corresponding to the element of type 2 stabilized by G_2. Then G_2 induces on the points of B the alternating group A_6 and P_2 is the full preimage in G_2 of the stabilizer in this action of a partition of B into three pairs, and one of these pairs corresponds to the element of type 3 stabilized by G_3.

LEMMA 4.1. *A completion of the amalgam $\mathscr{B} = \{P_1, P_2, P_3\}$ of the stabilizers in $G \simeq Fi_{22}$ of elements from a maximal flag of $\mathscr{P}_{22}$ contains the amalgam $\mathscr{A} = \{G_1, G_2, G_3\}$ corresponding to the action of G on $\mathscr{G}^T_{22}$.*

PROOF. Let P be a completion of $\mathscr{B}$. To prove the lemma it is sufficient to reconstruct in P a subgroup isomorphic to G_2 and having the prescribed intersections with $G_1 = P_1$ and $G_3 = P_3$. Since $\mathscr{B}$ is a subamalgam in $\mathscr{A}$, we can consider in P_2 a subgroup $E = O_2(G_2)$. Let $N_i = N_{P_i}(E)$ and $\bar{N}_i = N_i/E$ for $i = 1$, 2 and 3. The amalgam $\mathscr{D} = \{\bar{N}_1, \bar{N}_2, \bar{N}_3\}$ is a subamalgam in $G_2/E \simeq A_6 \times S_3$ and $\mathscr{D} = \{A_6 \times 2, S_4 \times S_3, S_4 \times S_3\}$, where one of the S_4's intersects the other one exactly in a Sylow 2-subgroup. It is easy to check that G_2/E is actually the unique completion of $\mathscr{D}$. Hence the subgroup in P generated by N_i's for $i = 1$, 2 and 3 is isomorphic to G_2 and by the construction it has the right intersections with G_1 and G_3. □

Let $\mathscr{P}_{23}$ be the minimal 2-local parabolic geometry of $G \simeq Fi_{23}$ from [22], for which we also adopt the reverse ordering of types. Let P_i be the stabilizer in G of the element of type i in a maximal flag of $\mathscr{P}_{23}$. Then $P_1 = G_1$, $P_3 = G_3$ and $P_4 = G_5$, while P_2 is a proper subgroup in $G_2 \simeq 2^{6+8}.(A_7 \times S_3)$, which is the full preimage of an $L_3(2)$-subgroup in the A_7-factor. By arguments similar to that in Lemma 4.1, one can prove the following:

LEMMA 4.2. *A completion of the amalgam $\mathscr{B} = \{P_1, P_2, P_3\}$ of the stabilizers in $G \simeq Fi_{23}$ of the elements of type* 1, 2 *and* 3 *in a maximal flag of $\mathscr{P}_{23}$ contains the amalgam $\mathscr{A} = \{G_1, G_2, G_3\}$ corresponding to the action of G on $\mathscr{G}^T_{23}$.*

Now, by Lemmas 4.1 and 4.2 and Corollary 3.2, we come to the following result, proved independently in [23, 24] and in [25].

PROPOSITION 4.3. *The minimal 2-local parabolic geometries of the groups Fi_{22} and Fi_{23} are simply connected.*

Now let us turn to the 2-local parabolic geometries of the largest Fischer group Fi_{24}. The maximal and minimal parabolic geometries were defined in [21] and [22] for the simple subgroup Fi'_{24}, but we prefer to deal with the Fischer group itself. Detailed information on the maximal parabolic geometry can be found in [12] and [27]. We start by describing the geometries in terms already familiar to us.

Let Ξ be the collinearity graph of $\mathscr{G}_{24}$. The elements of type 2 are identified with a certain class of triangles (actually with all the triangles) in Ξ. The group $G \simeq Fi_{24}$ acts naturally on Ξ, with the stabilizer $G(x)$ of a vertex x in G being (a conjugate of) $G_1 \simeq 2^{12}.M_{24}$. Let $S(x) = (P(x), B(x))$ be the corresponding Steiner system. The module $O_2(G(x))$ is the Golay cocode that is the $GF(2)$-space of all the subsets of $P(x)$ factorized over the submodule generated by all the blocks from $B(x)$. In particular, $O_2(G(x))$ contains a generating set of involutions identified with the point set $P(x)$. Let T be a 4-element subset of $P(x)$ (a tetrad). Recall that a certain element of type 3 in $\mathscr{G}_{24}$ (incident to x) can be identified with a subgraph on 27 vertices, which is the connected component containing x of the subgraph induced in Ξ by the vertices fixed by a subgroup (of order 2^4) generated by the points in T. In the Steiner system $S(x) \simeq S(5, 8, 24)$ the tetrad T determines a unique sextet; that is, a partition of $P(x)$ into 6 tetrads $T_1 = T, T_2, \ldots, T_6$ such that $T_i \cup T_j \in B(x)$ for all $1 \leqslant i \neq j \leqslant 6$. Since $O_2(G(x))$ is the Golay cocode, the product c of the points in T_i is independent of i and so c is in fact uniquely determined by the sextet. Such involutions defined for all the sextets in $S(x)$ form an orbit of $G(x)/O_2(G(x)) \simeq M_{24}$ in its action on $O_2(G(x))$ of length 1771 with the stabilizer $2^6.3 \cdot S_6$. These involutions will be said to be of the sextet type.

The centralizer $C = C_G(c)$ is isomorphic to $2^{1+12}.3 \cdot U_4(3).2^2$, its intersection with G_1 contains $O_{2,3}(C)$, and the image of the intersection in $C/O_{2,3}(C) \simeq U_4(3).2^2$ is a maximal subgroup of index 567 isomorphic to $2^4.S_6$. Suppose that the block B corresponding to the element y of type 2 stabilized by G_2 contains the tetrad T. By the property (iii) in Section 3, $C_{G_2}(c)$ induces S_3 on the elements of type 1 incident to y. By the property (i), c fixes exactly 30 vertices of Ξ adjacent to x which correspond to 15 blocks of $S(x)$ formed by the pairs of tetrads originating c. So we can formulate the following:

LEMMA 4.4. *The connected component $\Xi(c)$ containing x of the subgraph in Ξ induced by the vertices fixed by c is a graph on 567 vertices of valency 30. $C = C_G(c)$ induces on $\Xi(c)$ a primitive action of $U_4(3).2^2$.*

Notice that each of the 6 elements of type 3 in $\mathcal{G}_{24}$ incident to x corresponds to a 27-vertex subgraph of $\Xi(c)$ passing through x.

Recall that a trio in $S(5, 8, 24)$ is a partition of $P(x)$ into 3 blocks from $B(x)$. Having a sextet we naturally obtain 15 trios, each corresponding to a partition of the tetrads into pairs. Every trio can be obtained in this way from exactly 7 different sextets. This means that with a trio we can associate 7 involutions of sextet type from $O_2(G(x))$. It turns out that these involutions generate a subgroup E of order 2^3 and $N_G(E) \simeq 2^{3+12}.(S_6 \times L_3(2))$. Let $\Xi(E)$ be the connected component containing x of the subgraph induced by the vertices fixed by E. Of course, if the sextet and the trio are chosen consistently, then $\Xi(E)$ is a subgraph of $\Xi(c)$. On the other hand, $N_{G_1}(E)$ is the full preimage in $N_G(E)$ of a subgroup $S_4 \times 2$ in the S_6-factor. It can be checked using the properties (i)–(iii) that E fixes exactly 6 vertices adjacent to x which correspond to the blocks forming the trio.

LEMMA 4.5. *The connected component $\Xi(E)$ containing x of the subgraph in Ξ induced by the vertices fixed by E is a graph on 15 vertices of valency 6 on which $N_G(E)$ induces an action of S_6.*

Certainly, $\Xi(E)$ is the collinearity graph of the generalized quadrangle of order $(2,2)$.

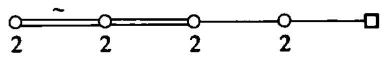
Now the maximal parabolic geometry $\mathcal{P}_{24}$ of $G \simeq Fi_{24}$ is a rank 4 geometry, the elements of types 1, 2, 3 and 4 of which are the vertices, the triangles and the images under G of the subgraphs $\Xi(E)$ and $\Xi(c)$. $\mathcal{P}_{24}$ is a locally truncated Tits geometry corresponding to the following diagram (the elements of type 5 exist only locally):

$\mathcal{P}_{24}$:

The residue of an element of type 1 is the geometry of octads (blocks), trios and sextets of $S(5, 8, 24)$ with the natural incidence relation. The residue of an element of type 4 is a GAB (a geometry which is almost a building) of $U_4(3)$, described in [11].

Starting with the graph defined in the similar way on the set of elements of type 1 in the geometry $3.\mathscr{G}_{24}$ acted on by $G \simeq 3 \cdot Fi_{24}$, we can apply the same construction procedure. Since the vertex stabilizer $G(x)$ is the same as above, we can choose an involution c and a 2^3-subgroup E. But now $C = C_G(c) \simeq 2^{1+12}.3^2 \cdot U_4(3).2^2$, where the factor $C/O_2(C)$ is a 'completely non-split' extension of $U_4(3).2^2$ by a group of order 9. This extension can be found as a subgroup of Co_1. In fact, $3 \cdot Fi_{24}$ is a subgroup in the Monster group M and $C_M(c) \simeq 2^{1+24}.Co_1$. Similarly, $N_G(E) \simeq 2^{3+12}.(3 \cdot S_6 \times L_3(2))$ and it is contained in a maximal 2-local subgroup of the Monster of the shape $2^{3+6+12+18}.(3 \cdot S_6 \times L_3(2))$.

Since $C_{G_1}(c)$ and $N_{G_1}(E)$ are the same as in the Fi_{24} case, we see that the subgraphs $\Xi(c)$ and $\Xi(E)$ are triple covers of their analogues from the Fi_{24} case. $C_G(c)$ and $N_G(E)$ induce on the corresponding subgraphs the actions isomorphic to $3 \cdot U_4(3).2^2$ and $3 \cdot S_6$, respectively. So in the case considered $\Xi(E)$ is the collinearity graph of the famous triple cover of the generalized quadrangle or order $(2, 2)$. This cover is denoted by the 'tilde diagram'. Thus we come to a geometry $3.\mathscr{P}_{24}$ acted on by $3 \cdot Fi_{24}$ and corresponding to the diagram:

$3.\mathscr{P}_{24}$: 

Notice that $3.\mathscr{P}_{24}$ is not a cover of $\mathscr{P}_{24}$ (it is only a 1-cover) and we hope that our notation will not be confusing.

By the construction, $\Xi(\tilde{c})$ still contains 6 subgraphs on 27 vertices, each realizing elements of type 3 in $3.\mathscr{G}_{24}$ incident to x.

THEOREM 4.6. *The geometries $\mathscr{P}_{24}$ and $3.\mathscr{P}_{24}$ are simply connected.*

PROOF. Let $\mathscr{P}$ be $\mathscr{P}_{24}$ or $3.\mathscr{P}_{24}$, let $\mathscr{G}$ be $\mathscr{G}_{24}^T$ or $3.\mathscr{G}_{24}^T$ and let $G \simeq Fi_{24}$ or $3 \cdot Fi_{24}$, respectively. Then $\mathscr{P}$ and $\mathscr{G}$ have the same sets of elements of type 1 and 2. So they share the collinearity graph Ξ. We can realize elements of both $\mathscr{P}$ and $\mathscr{G}$ by certain subgraphs of Ξ. Of course, the elements of types 1 and 2 are realized by the same subgraphs in both geometries, and a subgraph realizing an element of type 3 in $\mathscr{G}$ is contained in a subgraph realizing an element of type 4 in $\mathscr{P}$.

Let $\phi: \tilde{\mathscr{P}} \to \mathscr{P}$ be the universal covering of $\mathscr{P}$. Let $\tilde{\Xi}$ be the collinearity graph of $\tilde{\mathscr{P}}$ and let $\psi: \tilde{\Xi} \to \Xi$ be the induced covering of graphs. Let Δ be a subgraph in Ξ realizing an element of $\mathscr{P}$. Since ϕ is a covering, $\psi^{-1}(\Delta)$ is a disjoint union of subgraphs, each isomorphic to Δ. Because of the remark in the previous paragraph the same is true for every subgraph Δ realizing an element of $\mathscr{G}$. This means that ϕ induces a covering of $\mathscr{G}$. By Proposition 3.1, such a covering is either an isomorphism or the covering $3.\mathscr{G}_{24}^T \to \mathscr{G}_{24}^T$. But since $3.\mathscr{P}_{24}$ is not a cover of $\mathscr{P}_{24}$ the latter possibility cannot occur and the result follows. $\square$

We now have a standard implication of Theorem 4.6.

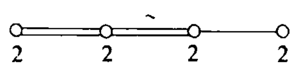
COROLLARY 4.7. *Let $\mathscr{P} \simeq \mathscr{P}_{24}$ or $3.\mathscr{P}_{24}$ and let G act flag-transitively on $\mathscr{P}$ (that is, $G \simeq Fi_{24}$, Fi_{24}', $3 \cdot Fi_{24}$ or $3 \cdot Fi_{24}'$). Let $\mathscr{C} = \{P_1, P_2, P_3, P_4\}$ be the amalgam of the* *stabilizers in G of the elements from a maximal flag in $\mathcal{P}$. Then G is the unique completion of $\mathcal{C}$.*

In the above notation, the residue of an element of type 1 is the maximal 2-local parabolic geometry of M_{24}, which is proved in [20] to be simply connected. This means that P_1 is the unique completion of the amalgam $\{P_1 \cap P_2, P_1 \cap P_3, P_1 \cap P_4\}$ and we have the following:

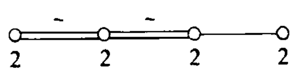
PROPOSITION 4.8. *The conclusion of Corollary 3.7 holds if we substitute $\mathcal{C}$ by $\{P_2, P_3, P_4\}$.*

Notice that up to conjugation in G the amalgam $\{P_2, P_3, P_4\}$ in Proposition 4.8 is uniquely determined by the conjugacy classes of P_i's under the condition that they have a Sylow 2-subgroup of G in common.

Now let us discuss the minimal parabolic geometries. Let $\mathcal{Q}_{24}$ be the minimal parabolic geometry of Fi'_{24} constructed in [22]. The group $G \simeq Fi_{24}$ acts on $\mathcal{Q}_{24}$. Let $Q_1, \ldots, Q_4$ be the stabilizers in G of the elements in a maximal flag of $\mathcal{Q}_{24}$. Then $Q_1 = P_1$ and $Q_4 = P_4$ while Q_2 and Q_3 are proper subgroups in P_2 and P_3, respectively (here the P_i's are as in Corollary 4.7). The subgroup Q_2 is the full preimage in $P_2 \simeq 2^{7+8}.(S_3 \times A_8)$ of a subgroup $2^3.L_3(2)$ in the A_8-factor and Q_3 is the full preimage in $P_2 \simeq 2^{3+12}.(S_6 \times L_3(2))$ of an S_4-subgroup in the $L_3(2)$-factor. The diagram of $\mathcal{Q}_{24}$ is the following (cf. [22]):

$\mathcal{Q}_{24}$: 

By taking analogous subgroups in the stabilizers in $G \simeq 3 \cdot Fi_{24}$ of elements from a maximal flag of $3.\mathcal{P}_{24}$, we obtain a geometry corresponding to the following diagram and acted on flag-transitively by $3 \cdot Fi_{24}$ and $3 \cdot Fi'_{24}$.

$3.\mathcal{Q}_{24}$: 

THEOREM 4.9. *The geometries $\mathcal{Q}_{24}$ and $3.\mathcal{Q}_{24}$ are simply connected.*

PROOF. Let $\mathcal{Q} \simeq \mathcal{Q}_{24}$ or $3.\mathcal{Q}_{24}$, $\mathcal{P} \simeq \mathcal{P}_{24}$ or $3.\mathcal{P}_{24}$ and $G \simeq Fi_{24}$ or $3 \cdot Fi_{24}$ respectively. Let $\{P_1, \ldots, P_4\}$ and $\{Q_1, \ldots, Q_4\}$ be the amalgams of stabilizers corresponding to the action of G on $\mathcal{Q}$ and $\mathcal{P}$, respectively. It is easy to check (compare Lemma 4.1) that for $i = 2$ and 3 the group P_i is the unique completion of the amalgam consisting of the subgroups $N_{Q_j}(O_2(P_i))$ for $j = 1, \ldots, 4$. The claim now follows from Corollary 4.7. □

Since the residue of an element of type 1 in $\mathcal{Q}_{24}$ is the minimal 2-local parabolic geometry of M_{24} , which is simply connected by [8], we have the following analogue of Proposition 4.8.

PROPOSITION 4.10. *Let $\{Q_2, Q_3, Q_4\}$ be the amalgam consisting of the stabilizers in*

$G \simeq F_{24}$ of the elements of type 2, 3 and 4 from a maximal flag in $\mathcal{Q}_{24}$. Then G is the unique completion of this amalgam.

5. Extended Dual Polar Spaces

The next class of geometries associated with the Fischer groups we will consider, are extensions of dual polar spaces of orthogonal and symplectic types. As above for $G \simeq Fi_{21+s}$, $1 \leq s \leq 3$ by G_i we denote the statilizer in G of an element of type i from a maximal flag of $\mathcal{G}_{24+s}$. In addition, for $s = 3$ by P_i we denote similar subgroups associated with the action of G on $\mathcal{P}_{24}$. We always assume that $P_1 = G_1$ and $P_2 = G_2$. A description of the extended dual polar spaces associated with the Fischer groups can be found in [29]. We describe them in slightly different terms below.

The group $G \simeq Fi_{24}$ contains a subgroup $H \simeq \Omega_{10}^-(2).2$. This subgroup, acting on the vertex set of Γ_{24} (that is, on the set D_{24} of 3-transpositions) has three orbits Ω_1, Ω_2 and Ω_3 with stabilizers isomorphic to $Sp_8(2) \times 2$, S_{12} and $2^{6+8}.(A_7 \times S_3)$, respectively (cf. [6]). The orbit Ω_1 consists of the transpositions contained in H. The action of H on Ω_3 preserves an imprimitivity system with blocks of size 8. Every such block is an element of type 2 in $\mathcal{G}_{24}$ and its stabilizer F in H is of the shape $2^{6+8}.(A_8 \times S_3)$. So F has index 2 in (a conjugate of) G_2. On the other hand, F is the stabilizer of a point in the dual polar space Π naturally associated with H.

Let Δ be a graph on the set of G-conjugates of H where two subgroups are adjacent if their intersection is a conjugate of F.

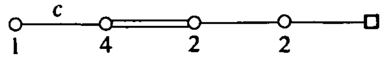
Then H is the stabilizer of a vertex x in the natural action of G on Δ. Moreover, the action of H on the set $\Delta(x)$ of vertices adjacent to x is similar to its action on the point set of the dual polar space Π. We can assume that for a certain vertex $y \in \Delta(x)$ the set-wise stabilizer of $\{x, y\}$ is $G_2 = P_2 \simeq 2^{7+8}.(A_8 \times S_3)$ and the element-wise stabilizer is $F \simeq 2^{6+8}.(A_8 \times S_3)$. Notice that the centres of $O_2(P_2)$ and $O_2(F)$ are the natural permutation module of A_8 and the even half of this module, respectively, both factorized over the subspace of constant functions.

Let p be a plane in Π which contains the point (corresponding to) y. The stabilizer $H(p)$ of p in H is the centralizer in H of an involution c and $H(p) \simeq 2^{1+12}.(S_3 \times U_4(2).2)$. The subgroup $O_{\{2,3\}}(H(p))$ is the kernel of the action of $H(p)$ on the set of 45 points (vertices in $\Delta(x)$) incident to p and the induced action is similar to the action of $U_4(2).2$ on the set of lines of the generalized quadrangle of order $(2, 4)$. It is easy to see that c is contained in the centre of $O_2(F)$ and hence $C_{G_2}(c) \neq C_F(c)$. This implies, in particular, that c is a central involution in G and hence $C_G(c)$ coincides with (a G-conjugate of) $P_4 \simeq 2^{1+12}.3 \cdot U_4(3).2^2$ and contains $H(p)$ as a subgroup of index 126.

Let $\Delta(c)$ be the subgraph induced by the images under P_4 of the vertex x. By the previous paragraph we see that it contains 126 vertices; P_4 induces on $\Delta(c)$ a rank 3 action of $U_4(3).2^2$ with the subdegrees 1, 45 and 80 (cf. [3, p. 399]) and the subgraph is non-empty. Since $C_H(c) = H(p)$ does not have an orbit of length 80 on $\Delta(x)$ we conclude that $\Delta(c)$ is of valency 45 and locally it is the line graph of the generalized quadrangle of order $(2, 4)$.

Now we see that two vertices in $\Delta(x)$ are adjacent if they correspond to collinear points in Π. Let l be a line in Π incident to y and p. The points in $\Delta(x)$ incident to l together with x form a clique K of size 6 which contains y and is contained in $\Delta(c)$. The stabilizer of K in H is of the form $2^{3+12}.(S_5 \times L_3(2))$ and it induces on K an action of S_5. It is easy to see that the full stabilizer of K in G acts transitively on the vertices of K. So this stabilizer must be of the shape $2^{3+12}.(S_6 \times L_3(2))$ and we immediately identify it with (a conjugate of) P_3.

Now we define a geometry $\mathcal{H}_{24}$ the elements of which are the vertices and the edges of Δ and also the images under G of the subgraphs K and $\Delta(c)$. As usual, we define the incidence by inclusion. Then $\mathcal{H}_{24}$ corresponds to the following locally truncated diagram:

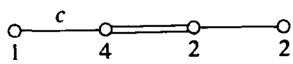
$\mathcal{H}_{24}$: 

The elements of type 5 exist only locally. In fact, suppose $\mathcal{H}_{24}$ is a truncation of a rank 5 c-extension of Π, admitting G as an automorphism group. Then the stabilizer in G of an element u of type 4 must induce S_3 on the set of elements of type 5 incident to u. But $G(u) = P_4$ does not have a factor group isomorphic to S_3 (since the extension of $U_4(3)$ by the centre of order 3 is non-split).

As observed, for $i = 2$, 3 and 4 the stabilizer in G of an element of type i from a maximal flag in $\mathcal{H}_{24}$ is a G-conjugate of P_i. Moreover, these stabilizers contain a common Sylow 2-subgroup. So, by Proposition 4.8 and the remark after it, we have the following:

Proposition 5.1. *The geometry $\mathcal{H}_{24}$ is simply connected.*

There is a more or less standard way to get rid of the fake node in the diagram by considering the 'minimal parabolics'. Let $H_1 = H$, $H_2 = P_2$, $H_3 = P_3$ and $H_4 = P_4$ be the stabilizers in G of elements from a maximal flag in $\mathcal{H}_{24}$. Define subgroups $K_i \leqslant H_i$ by the following rule: K_1 is the stabilizer of a hyperplane in $\Delta(x)$ containing the plane stabilized by $H_1 \cap H_4$; $K_4 = H_4$, $K_2 = Q_2$ and $K_3 = Q_3$, where the Q_i's are the stabilizers in G of elements from the minimal parabolic geometry $\mathcal{Q}_{24}$. Let K_{24} be a geometry the elements of type i of which are the cosets of K_i in G, $1 \leqslant i \leqslant 4$; two elements are incident if they have a non-empty intersection. Then $\mathcal{K}_{24}$ belongs to the following diagram:

$\mathcal{K}_{24}$: 

There is a combinatorial way to construct $\mathcal{K}_{24}$ from $\mathcal{H}_{24}$. Let $\{x, y\}$ be an edge of Δ. The subgraphs in Δ corresponding to the elements of $\mathcal{H}_{24}$ incident to $\{x, y\}$ define a bijection $\phi_{x,y}$ from the set of hyperplanes in $\Delta(x)$ containing y onto the set of hyperplanes in $\Delta(y)$ containing x. This bijection is the unique one commuting with the action of the stabilizer of $\{x, y\}$ in G.

Let Φ be a graph the vertices of which are the pairs (x, h), where x is a vertex of Δ and h is a hyperplane in $\Delta(x)$. Two vertices (x, h) and (y, k) are adjacent if x and y are adjacent in Δ, $y \in h$, $x \in k$ and $\phi_{x,y}(h) = k$. Then Φ is the collinearity graph of $\mathcal{K}_{24}$. The elements of type 3 and 4 in $\mathcal{K}_{24}$ are realized respectively by complete 6-vertex subgraphs and certain subgraphs on 378 vertices in Φ. A subgraph realizing an element of type 4 is a proper triple cover of a subgraph in Δ realizing an element of type 4 in $\mathcal{H}_{24}$. The cover is acted on by a non-split extension of $U_4(3)$ by a normal subgroup of order 3. A description of this cover can be found in [3, p. 399].

Proposition 4.10 and the construction of $\mathcal{K}_{24}$ in terms of the subamalgam immediately imply the following.

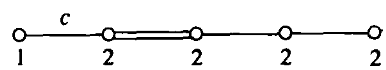
Proposition 5.2. *The geometry $\mathcal{K}_{24}$ is simply connected.*

Now we are going to obtain a certain geometry for Fi_{23} as a subgeometry in $\mathcal{H}_{24}$ on the elements fixed by a certain involution from Fi_{24}. The above description of the orbits of H on the set D_{24} of 3-transpositions shows that the intersection of D_{24} and H forms a conjugacy class in H. The latter is the class of central involutions in H and if t

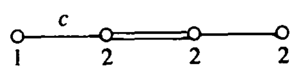
is an involution from this class then $C_H(t) \simeq \langle t\rangle \times Sp_8(2)$. This means in particular that $C_G(t) \simeq \langle t\rangle \times Fi_{22}$ acts transitively on the set $\Psi = \Psi(t)$ of vertices in Δ fixed by t and the action is similar to the action of Fi_{23} on the cosets of $Sp_8(2)$. Let Ψ also denote the subgraph in Δ induced by the vertices fixed by t. As above we assume that the elements of type 1, 2, 3 and 4 in $\mathcal{H}_{24}$ are the vertices, the edges, the 6-cliques and certain 126-vertex subgraphs in Δ, respectively. In the dual polar space Π associated with x the elements fixed by the involution t form the dual polar space naturally associated with $C_H(t)/\langle t\rangle \simeq Sp_8(2)$. This means that the intersection with Ψ of an element of type 3, if non-empty, must be a 4-clique. Similarly, if an element of type 4 properly intersects Ψ then the intersection is a graph of valency 15 which is locally the collinearity graph of the generalized quadrangle of order $(2, 2)$. On the other hand, this intersection is the fixed point set of an involution from $U_4(3).2^2$ acting on the vertices of Δ incident to the element of type 4. This implies that the intersection is a graph on 36 vertices with the automorphism group isomorphic to $U_4(2).2$.

Let $\mathcal{H}_{23}^T$ be a rank 4 geometry the elements of which of type 1, 2, 3 and 4 are the vertices of Ψ, the edges of Ψ and the non-empty intersections with Ψ of elements of type 3 and 4 in $\mathcal{H}_{24}$. Then $\mathcal{H}_{23}^T$ is a c-extension of the dual polar space of symplectic type, truncated by the hyperplanes. In the considered case we can recover the hyperplanes by defining elements of type 5 globally.

Put $G \simeq Fi_{23}$ and let $S \simeq Sp_8(2)$ be the stabilizer in G of a vertex $x \in \Psi$. The set $\Psi(x)$ of neighbours of x in Ψ is naturally identified with the point set of the dual polar space associated with S. It is known (cf. [6]) that the intersection of S with the class D_{23} of 3-transpositions in G forms a conjugacy class of (central) involutions in S. If s is an involution from the intersection then $C_S(s) \simeq 2^7.Sp_6(2)$. In particular, $C_G(s) \simeq 2 \cdot F_{22}$ acts transitively on the set $\Theta(s)$ of vertices in Ψ fixed by s. On the other hand, $C_S(s)$ is the stabilizer in S of a hyperplane from $\Psi(x)$ and the elements of the dual polar space incident to the hyperplane are exactly those fixed by s. We define the elements of type 5 to be the images under G of the subgraph induced by $\Theta(s)$ with the incidence relation defined by the inclusion. The resulting rank 5 geometry $\mathcal{H}_{23}$ belongs to the following diagram:

$\mathcal{H}_{23}$:


By the construction, $\mathcal{H}_{23}$ is acted on flag-transitively by Fi_{23}. The residue of an element of type 5 will be denoted by $\mathcal{H}_{22}$ and, of course, it belongs to the following diagram:

$\mathcal{H}_{22}$:


It is easy to see that $\mathcal{H}_{22}$ admits the action of the full automorphism group $Fi_{22}.2$ of Fi_{22}.

The geometry $\mathcal{H}_{22}$ possesses a triple cover which can be described as follows. Consider the action of $L \simeq Fi_{22}.2$ on $\mathcal{H}_{22}$ and let L_i be the stabilizer in L of the element of type i in a maximal flag of $\mathcal{H}_{22}$. In particular, we have $L_1 \simeq 2^7.Sp_6(2)$ and $L_4 \simeq (2 \times 2^{1+8}).2.U_4(2).2$, the latter being a conjugate of G_3. It is clear that there is a Sylow 2-subgroup R of L which contains $O_2(L_i)$ for $1 \leq i \leq 4$, and it is not difficult to check that the following two conditions hold for every $1 \leq i \leq 4$: (a) $N_L(O_2(L_i)) = L_i$; (b) $O_2(L_i) \cap L' \neq O_2(L_i)$. Here $L' \simeq Fi_{22}$ is the derived group of L. The 3-part of the Schur multiplier of Fi_{22} is of order 3 and in the unique non-split extension $\tilde{L} \simeq 3 \cdot L$ the normal 3-subgroup is not central. Let $\tilde{R}$ be a Sylow 2-subgroup in the full preimage of R in $\tilde{L}$. The natural homomorphism of $\tilde{L}$ onto L induces an isomorphism ϕ of $\tilde{R}$ onto

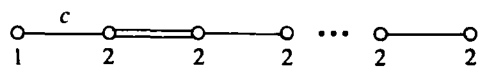
R. Let $R_i = \phi^{-1}(O_2(L_i))$ and $\tilde{L}_i = N_{\tilde{L}}(R_i)$ for $1 \leq i \leq 4$. By the conditions (a) and (b) the amalgam formed by $\tilde{L}_i$'s in $\tilde{L}$ is isomorphic to that formed by L_i's in L. Let $3.\mathcal{H}_{22}$ be a geometry the elements of type i of which are the cosets of $\tilde{L}_i$ in $\tilde{L}$; two elements are incident if they have a non-empty intersection. By the construction $3.\mathcal{H}_{22}$ is a proper cover of $\mathcal{H}_{22}$ and, in particular, these two geometries correspond to the same diagram. Notice that $\mathcal{H}_{22}$ can also be found as a subgeometry in $\mathcal{H}_{24}$ [18].

In the next section we will prove a characterization result which implies the following:

PROPOSITION 5.3. *The geometry $\mathcal{H}_{23}$ is simply connected and $3.\mathcal{H}_{22}$ is the universal cover of $\mathcal{H}_{22}$.*

6. A CHARACTERIZATION

In this section we consider flag-transitive c-extensions of the dual polar spaces of symplectic type defined over $GF(2)$. So a rank $n+1$ geometry under consideration belongs to the diagram

$c.C_n(2)$: 

and the residue of an element of the leftmost type is the dual polar space with the automorphism group isomorphic to $Sp_{2n}(2)$.

Let $\mathcal{G}$ be a $c.C_n(2)$-geometry, let G be a flag-transitive automorphism group of $\mathcal{G}$ and let Γ be the collinearity graph of $\mathcal{G}$.

For the case $n = 2$ the complete classification was obtained in [5] (see also [3, p. 258] and [28]).

PROPOSITION 6.1. *Suppose that in the above notation $n = 2$. Then one of the following holds*:

(*i*) *Γ is the Taylor graph on 32 vertices and $2^5.A_6 \leq G \leq 2^5.S_6$;*

(*ii*) *$\mathcal{G}$ is a folding of the geometry from (i), Γ is the complete graph on 16 vertices and $2^4.A_6 \leq G \leq 2^4.S_6$;*

(*iii*) *Γ is the complement of the Johnson graph $J(8, 2)$, $A_8 \leq G \leq S_8$; and*

(*iv*) *Γ is a graph on 36 vertices and $U_4(2) \leq G \leq U_4(2).2$.*

In the cases (i), (iii) and (iv) the elements of the geometry are the vertices, the edges and all the triangles in Γ. In the case (ii) only one of the two G-orbits on the triangles corresponds to elements of type 3.

So the flag-transitive $c.C_n(2)$-geometries can be divided into four classes depending on the structure of $c.C_2(2)$-residues (the cases (i)–(iv) in Proposition 6.1).

Let us mention some examples corresponding to the case (i). Let $\Pi = \Pi(n)$ denote the rank n dual polar space of symplectic type defined over $GF(2)$. Suppose that there exists a group R and a subset $S \subset R$ such that (1) S generates R; (2) S consists of involutions; (3) there is a bijection ϕ from the point set of Π onto S such that for every line l in Π the images under ϕ of the points on l product to the identity element. Let t be an element of type i in Π. It follows from the above conditions that the images under ϕ of the points incident to t generate in R an elementary abelian subgroup of order 2^i. So we come to a bijection ϕ of the element set of Π into the set of elementary abelian 2-subgroups in R which is a representation of Π (cf. [9]). If R is abelian then ϕ is an embedding of Π into the projective geometry formed by the proper subgroups of R. We can associate with R a $c.C_n(2)$-geometry the elements of which are the elements of R and the (right) cosets in R of all the images under ϕ of elements of Π. The incidence is defined by the inclusion. So as soon as we have a representation of Π (non-necessary abelian), we can construct a $c.C_n(2)$-geometry the $c.C_2(2)$-residues of which correspond to case (i) or (ii). It is well known that the spin module of $Sp_{2n}(2)$ supports an abelian representation of $\Pi(n)$. In particular, $c.C_n(2)$-geometries exist for all n. The universal abelian representation of $\Pi(n)$ is larger and its dimension was conjectured in [2] to be $(2^n+1)(2^{n-1}+1)/3$. In [28] the universal representation group of $\Pi(3)$ was proved to be $2.(2^{1+8}_{+} \times 2^6)$. In fact [28] contains the complete classification of flag-transitive $c.C_3(2)$-geometries the $c.C_2(2)$-residues of which correspond to case (i). Besides the geometries related to representations of $\Pi(3)$, there is exactly one other simply connected example with the automorphism group isomorphic to $Sp_8(2)$.

No examples corresponding to case (iii) in Proposition 6.1 are known to me, while the geometries $\mathscr{H}_{22}$, $3.\mathscr{H}_{22}$ and $\mathscr{H}_{23}$ from the previous section correspond to case (iv). From now on we will deal with the latter case only.

Let $\mathscr{G} = \mathscr{G}(U_4(2))$ be the $c.C_2(2)$-geometry corresponding to case (iv) in Proposition 6.1. The group $U_4(2).2$ is the Weyl group of type E_6. The corresponding 6-dimensional root space contains exactly 36 one-dimensional subspaces the vectors of which are multiples of roots. These subspaces are the vertices of the collinearity graph Γ of $\mathscr{G}$. Two subspaces are adjacent if they are perpendicular. Γ is strongly regular with the parameters $v=36$, $k=15$ and $\lambda=6$, and locally it is the collinearity graph of the generalized quadrangle of order $(2,2)$. So the elements of type 1, 2 and 3 in $\mathscr{G}$ can be identified with the vertices, edges and maximal cliques (of size 4) in Γ. The description of $\mathscr{G}$ in terms of the E_6-root system enables one to check directly the properties presented below.

Let u be an element of type 3 in $\mathscr{G}$. Then the elements of types 1 and 2 incident to u are the vertices and the edges of a 4-clique in Γ. Two disjoint edges in this clique will be called parallel. Let $\Lambda = \Lambda(\mathscr{G})$ be a graph on the edges of Γ in which two edges are adjacent if they are parallel inside a certain 4-clique. In other terms, Λ is a graph on the set of elements of type 2 in $\mathscr{G}$, where two elements are adjacent if they are incident to a common element of type 3 but not to a common element of type 1. It is straightforward that Λ is of valency 3. A more detailed analysis shows the following:

LEMMA 6.2. *Let $\Lambda = \Lambda(\mathscr{G}(U_4(2)))$. Then Λ is a disjoint union of* 27 *copies of the Petersen graph. Two elements from the same connected component are never incident to a common element of type* 1.

Let Σ be a connected component of Λ, let G be a flag-transitive automorphism group of $\mathscr{G}$ (that is, $\mathscr{G} \simeq U_4(2)$ or $U_4(2).2$) and let H be the set-wise stabilizer of Σ in G. Let K_0 denote the element-wise stabilizer in G of the vertices of Σ.

LEMMA 6.3. *In the above notation $H/K_0 \simeq A_5$ or S_5; $K_0 \simeq 2^4$ and it acts faithfully on the set of elements of type* 1 *incident to elements from Σ.*

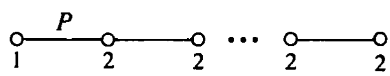
Every edge e of Λ determines a 4-clique in Γ (that is, an element of type 3 in $\mathscr{G}$), which we will denote by $\psi(e)$. By Lemma 6.2, a connected component Σ of Λ is isomorphic to the Petersen graph. There is a unique equivalence relation on the set of edges of the Petersen graph stable under its automorphism group. This is the antipodal relation on the line graph and has classes of size 3. If e and f are equivalent edges in Σ, then the elements $\psi(e)$ and $\psi(f)$ will be called parallel. Let u be an element of type 3 in $\mathscr{G}$. Then $\psi^{-1}(u)$ consists of three edges in different connected components of Λ. Each of the edges determines a pair of elements of type 3 different from u and parallel to u. It can be checked that this pair is independent on the choice of an edge in $\psi^{-1}(u)$. So we have the following:

LEMMA 6.4. *Every element of type* 3 *in* $\mathcal{G}(U_4(2))$ *is parallel to exactly two other such elements.*

The following property of the geometry $\mathcal{G}(U_4(2))$ can be checked directly.

PROPOSITION 6.5. *Let* $\mathcal{G} = \mathcal{G}(U_4(2))$. *Let* $\mathcal{H}$ *be a rank* 2 *geometry the elements of type* 1 *of which are the connected components of* $\Lambda(\mathcal{G})$ *and the elements of type* 2 *of which are the parallel classes of elements of type* 3 *in* $\mathcal{G}$. *Then, with respect to the incidence relation induced by that in* $\mathcal{G}$, *the geometry* $\mathcal{H}$ *is isomorphic to the generalized quadrangle of order* (2, 4) *associated with* $U_4(2)$.

Now let $\mathcal{G}$ be a flag-transitive $c.C_n(2)$-geometry for $n \geq 3$ with the $c.C_2(2)$-residues isomorphic to $\mathcal{G}(U_4(2))$. Let $\Lambda = \Lambda(\mathcal{G})$ be a graph on the set of elements of type 2 in $\mathcal{G}$, in which two elements are adjacent if they are incident to a common element of type 3 but not to a common element of type 1. Let Σ be a connected component of Λ. Define a geometry $\mathcal{P}$ the elements of type 1 and 2 of which are the vertices and the edges of Σ, respectively; an element of type $i \geq 3$ in $\mathcal{P}$ is the connected component of the subgraph in Σ (if non-empty) induced by the vertices incident to an element of type $i+1$ in $\mathcal{G}$. The incidence is defined by the inclusion. By the construction, the residue in $\mathcal{P}$ of an element of type 1 is isomorphic to the residue in $\mathcal{G}$ of an element of type 2; that is, to a rank n projective space over $GF(2)$. This and Lemma 6.2 imply that $\mathcal{P}$ belongs to the following diagram:

$\mathcal{P}_n$: 

where the leftmost edge is the geometry of vertices and edges of the Petersen graph. Geometries corresponding to diagrams of this shape are called P-geometries. So we have the following:

PROPOSITION 6.6. *Let* $\mathcal{G}$ *be a flag-transitive* $c.C_n(2)$*-geometry the* $c.C_2(2)$*-residues of which are isomorphic to* $\mathcal{G}(U_4(2))$. *Then the above-defined geometry* $\mathcal{P}$, *associated with a connected component of the graph* $\Lambda(\mathcal{G})$, *is a flag-transitive* P*-geometry of rank* n.

The flag-transitive P-geometries have been completely classified by S. V. Shpectorov and the present author.

PROPOSITION 6.7. [10]. *Up to isomorphism there are exactly* 8 *flag-transitive* P*-geometries of rank at least* 3: $\mathcal{G}(M_{22})$ *and* $\mathcal{G}(3 \cdot M_{22})$ *of rank* 3; $\mathcal{G}(M_{23})$, $\mathcal{G}(Co_2)$, $\mathcal{G}(3^{23} \cdot Co_2)$ *and* $\mathcal{G}(J_4)$ *of rank* 4; $\mathcal{G}(F_2)$ *and* $\mathcal{G}(3^{4371} \cdot F_2)$ *of rank* 5.

In the above proposition, $\mathcal{G}(F)$ denotes the P-geometry the minimal flag-transitive automorphism group of which is isomorphic to F. The full automorphism group is twice as large ($M_{22}.2$ and $3 \cdot M_{22}.2$) in the rank 3 case. In other cases it coincides with F.

By Propositions 6.6 and 6.7 we immediately see that flag-transitive $c.C_n(2)$-geometries with the $c.C_2(2)$-residues isomorphic to $\mathcal{G}(U_4(2))$ exist only for $n \leq 5$. In fact, we can classify these geometries completely.

Let us extend the parallelism relation defined on the elements of type 3 in a $c.C_2(2)$-residue. We say that two elements of type 3 in $\mathcal{G}$ are parallel if they are incident to a common element of type 4 and are parallel in the corresponding $c.C_2(2)$-residue (isomorphic to $\mathcal{G}(U_4(2))$).

Let u be an element of type 3 in $\mathcal{G}$. The elements of type 2 incident to u induce in Λ a subgraph consisting of three disjoint edges. We know that each of these edges determines u inside a $c.C_2(2)$-residue. Since the residue of a flag of type $\{1,2\}$ in $\mathcal{G}$ is a projective space, one can see that an edge of Λ still determines a unique element of type 3 in the whole geometry. So there is a well defined mapping ψ from the edge set of Λ onto the set of elements of type 3 in $\mathcal{G}$, and the preimage of an element consists of three disjoint edges.

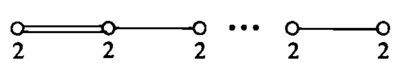
Let $e = \{y, z\}$ be an edge of Λ such that $\psi(e) = u$. Let Σ be the connected component of Λ containing e, and let $\mathcal{P}$ be the P-geometry associated with Σ (compare with Proposition 6.6). So e is an element of type 2 in $\mathcal{P}$. Let w be an element of type 4 in $\mathcal{G}$ incident to u. Let v be an element of type 3 also incident to w which is parallel to u in the $c.C_2(2)$-residue associated with w. Then w is an element of type 3 in $\mathcal{P}$. The elements of type 1 and 2 in $\mathcal{P}$ incident to w form a Petersen graph geometry and e is one of the edges. By Lemma 6.4, there is an edge f equivalent to e in the Petersen graph such that $\psi(f) = v$. On the other hand, for every edge f equivalent to e in a residual Petersen graph geometry, the element $\psi(f)$ is parallel to u.

Let M be a graph on the set of elements of type 3 in $\mathcal{G}$ in which two elements are adjacent if they are parallel. Let Θ be the connected component of M containing u. By the above paragraph we have the following:

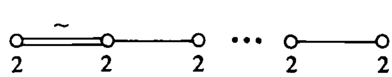
LEMMA 6.8. *Let Ξ be the connected component containing e of the graph on the set of elements of type 2 in $\mathcal{P}$, where two elements are adjacent if they are incident to a common element of type 3 and if in the corresponding residual Petersen graph geometry they are equivalent edges. Then ψ establishes an isomorphism of Ξ onto Θ.*

The graph on the set of elements of tpe 2 in a P-geometry defined as in Lemma 6.7 has already appeared in the theory of P-geometries. With a connected component Ψ of this graph we can associate a subgeometry $\mathcal{S}$ in $\mathcal{P}$ by the following rule. The elements of type 1 in $\mathcal{S}$ are the vertices of Ξ (elements of type 2 in $\mathcal{P}$) and an element of type $i \geq 2$ is a connected component of the subgraph (if non-empty) induced by the vertices of Ξ incident to an element of type $i+1$ in $\mathcal{P}$.

PROPOSITION 6.9. [10]. *Let $\mathcal{P}$ be a flag-transitive P-geometry of rank $n \geq 3$ and let $\mathcal{S}$ be the above defined subgeometry in $\mathcal{P}$ of rank $n-1$. Then $\mathcal{S}$ belongs either to the diagram*



or to the diagram



depending on whether the rank 3 residual P-geometries are isomorphic to $\mathcal{G}(M_{22})$ or to $\mathcal{G}(3 \cdot M_{22})$. The stabilizer of $\mathcal{S}$ in the full automorphism group of $\mathcal{P}$ is isomorphic to: $2^4.S_6$ for $\mathcal{G}(M_{22})$; $2^4.3 \cdot S_6$ for $\mathcal{G}(3 \cdot M_{22})$; $2^4.A_7$ for $\mathcal{G}(M_{23})$; $2^{1+8}.Sp_6(2)$ for $\mathcal{G}(Co_2)$; $2^{1+8}.3^7 \cdot Sp_6(2)$ for $\mathcal{G}(3^{23} \cdot Co_2)$; $2^{9+16}.Sp_8(2)$ for $\mathcal{G}(F_2)$ and $2^{9+16}.3^{35} \cdot Sp_8(2)$ for $\mathcal{G}(3^{4371} \cdot F_2)$.

Let Σ be a connected component of Λ and let H be the set-wise stabilizer of Σ in G. Let K_0 be the element-wise stabilizer in G of the elements of type 2 contained in Σ and let K_1 be the element-wise stabilizer in K_0 of all elements of type 1 incident to elements from Σ. Clearly, both K_0 and K_1 are normal subgroups in H. The following result follows from Proposition 6.6.

LEMMA 6.10. *In the above notation, H/K_0 is a flag-transitive automorphism group of the P-geometry $\mathcal{P}$ associated with Σ.*

By Lemma 6.10 and Proposition 6.7, $H/K_0 \simeq Aut(\mathcal{P})$ *or* $Aut(\mathcal{P})'$ and hence this section of H is known in a sense. Let us specify in terms of $\mathcal{P}$ the section K_0/K_1. If $K_1 \neq K_0$, then for every vertex of Σ (an element of type 2 in $\mathcal{G}$) the subgroup K_0 permutes the pair of elements of type 1 incident to x. In particular, K_0/K_1 is an elementary abelian 2-group which can be considered as a module for H/K_0. Let W be the module dual to K_0/K_1 for which we adopt the additive notation. Then, by the above remark, there is a mapping ϕ from the set of elements of type 1 in $\mathcal{P}$ (that is, from the vertex set of Σ) into W. Moreover, since K_0/K_1 is faithful on the set of elements of type 1 incident to elements from Σ, W is generated by the images of ϕ. Since we know explicitly the $c.C_2(2)$-residues, we are able to find certain relations on the images. By Lemma 6.3, the images under ϕ of the elements incident to an element of type 3 in $\mathcal{P}$ (there are 10 such elements) generate a 4-dimensional module. The structure of this 4-dimensional module implies the following:

LEMMA 6.11. *In the above notation, let u and v be incident elements of type* 1 *and* 3 *in the P-geometry $\mathcal{P}$. Let w_1, w_2 and w_3 be the elements of type* 2 *which are incident to both u and v. Let z_i be the element of type* 1 *incident to w_i and different from u for $1 \leq i \leq 3$. Then $\phi(z_1) + \phi(z_2) + \phi(z_3) = 0$.*

Modules generated by involutions indexed by elements of type 1 in a P-geometry and satisfying the 3-term relations from Lemma 6.11 have already appeared in [9] under the name of the 'derived modules'. It is clear how to define the universal derived module of a P-geometry (which actually might collapse).

PROPOSITION 6.12. *Let $\mathcal{P}$ be a flag-transitive P-geometry and $U = U(\mathcal{P})$ be the universal derived module of $\mathcal{P}$. Then:*
(i) if $\mathcal{P} \simeq \mathcal{G}(M_{22})$ or $\mathcal{G}(3 \cdot M_{22})$, then U is the irreducible 10*-dimensional section of the Golay code considered as an M_{22}-module;*
(ii) if $\mathcal{P} \simeq \mathcal{G}(M_{23})$, then U is the irreducible 11*-dimensional section of the Golay code considered as an M_{23}-module;*
(iii) if $\mathcal{P} \simeq \mathcal{G}(Co_2)$, $\mathcal{G}(3^{23} \cdot Co_2)$, $\mathcal{G}(F_2)$ or $\mathcal{G}(3^{4371} \cdot F_2)$, then U collapses.

PROOF. The universal derived module of $\mathcal{G}(M_{22})$ was determined in [9] and exactly the same arguments work for the $\mathcal{G}(3 \cdot M_{22})$ case. Using the fact that $\mathcal{G}(M_{22})$ is a residue of $\mathcal{G}(M_{23})$ it is easy to identify the universal derived module of the latter geometry. Also in [9] the triviality of the derived modules of $\mathcal{G}(Co_2)$ and $\mathcal{G}(3^{23} \cdot Co_2)$ was shown. Since these two geometries are residues in $\mathcal{G}(F_2)$ and $\mathcal{G}(3^{4371} \cdot F_2)$, respectively, for the latter two geometries the derived modules collapse as well. □

To the best of the author's knowledge, the determination problem of the universal derived module for $\mathcal{G}(J_4)$ has never been considered. From the information given in [6] one can see that there exists a 112-dimensional derived module of this geometry.

The residue in $\mathcal{G}$ of an element of type 1 is the rank n (dual) polar space of symplectic type defined over $GF(2)$. The stabilizer in G of such an element induces on the residual polar space a flag-transitive action. It is well known that such an action must be isomorphic to $Sp_{2n}(2)$. On the other hand, the automorphism group of a P-geometry of rank n does not involve $Sp_{2n}(2)$ (compare with [6]). This implies the following:

LEMMA 6.13. *The parallelism graph Λ on the set of elements of type 2 in $\mathscr{G}$ is disconnected.*

Now, by Lemmas 6.8 and 6.13, the parallelism graph M on the set of elements of type 3 in $\mathscr{G}$ is also disconnected.

A flag-transitive automorphism group of $\mathscr{G}(U_4(2))$ acts primitively both on the point and the line sets of the generalized quadrangle of order $(2,4)$. In view of Proposition 6.5 and Lemma 6.13 this implies the following:

LEMMA 6.14. *Let u be an element of type 4 in $\mathscr{G}$. Let x and y be elements both of type 2 or 3 incident to u. Then x and y are in the same connected component of Λ or M iff they are in the same component of the subgraph induced by the elements incident to u.*

Now we proceed to characterization of the geometries.

PROPOSITION 6.15. *Let $\mathscr{G}$ be a flag-transitive $c.C_3(2)$-geometry all of the $c.C_2(2)$-residues of which are isomorphic to $\mathscr{G}(U_4(2))$. Then $\mathscr{G}$ is isomorphic to $\mathscr{H}_{22}$ or $3.\mathscr{H}_{22}$.*

PROOF. Let $\mathscr{P}$ be the P-geometry associated with a connected component Σ of Λ. By Proposition 6.7, $\mathscr{P}$ is isomorphic either to $\mathscr{G}(M_{22})$ or to $\mathscr{G}(3.M_{22})$. The arguments for these two cases are slightly different.

Suppose that $\mathscr{P}=\mathscr{G}(M_{22})$. Define a rank 3 geometry $\mathscr{F}$ the elements of type 1 of which are the connected components of Λ, the elements of type 2 of which are the connected components of M and the elements of type 3 of which are all the elements of type 4 in $\mathscr{G}$. We assume that the incidence relation in $\mathscr{F}$ is induced by that in $\mathscr{G}$. By the construction, $\mathscr{F}$ belongs to a string diagram. By Lemma 6.14 and Proposition 6.5, the residue in $\mathscr{F}$ of an element of type 3 is the generalized quadrangle of order $(2,4)$. Consider the residue of an element of type 1 that is of a connected component Σ of Λ. By definition, the residue of Σ in $\mathscr{F}$ consists of all the subgeometries $\mathscr{S}$, as in Proposition 6.9, and all the elements of type 3 in $\mathscr{P}$. The geometry $\mathscr{G}(M_{22})$ possesses a description in terms of the Steiner system $S(3,6,22)$. In these terms the elements of type 3 are the 2-element subsets of points and the subgeometries are the blocks of the system with the natural incidence relation. This means that the diagram of $\mathscr{F}$ coincides with that of $\mathscr{G}_{22}^{T}$. Now we are going to prove that these two geometries are in fact isomorphic.

Let Δ be the collinearity graph of $\mathscr{F}$. Let H be the stabilizer in G of a connected component Σ of Λ. Then H is the stabilizer of a vertex, say x, in the natural action of G on Δ. It is clear that the subgroup K_1 coincides with the kernel of the action of H on the set $\Delta(x)$ of vertices adjacent to x in Δ. Suppose that $K_0=K_1$. Then, by Lemma 6.10, H induces on $\Delta(x)$ an action of M_{22} or $M_{22}.2$. There are 77 elements of type 2 incident to x, and H induces on these elements a primitive action of M_{22} or $M_{22}.2$ on the cosets of $2^4.A_6$ or $2^4.S_6$, respectively. There are 154 pairs (y,z) such that y is an element of type 2 incident to x and z is an element of type 1 incident to y and different from x. H must act transitively on the set of these pairs. Hence H induces on this set the action of $M_{22}.2$ on the cosets of $2^4.A_6$ (since M_{22} has no transitive representations of degree 154). Now, if $2^4.A_6<F<M_{22}.2$ then $F\simeq 2^4.S_6$ or M_{22}, and this immediately implies that any two elements of type 1 in $\mathscr{F}$ are incident to at most one common

element of type 2. In particular, $|\Delta(x)| = 154$, and the action of H on $\Delta(x)$ is similar to the action of $M_{22}.2$ on the cosets of $2^4.A_6$. Let u be an element of type 3 in $\mathcal{F}$ incident to x and let Ξ be the subgraph in Δ induces by the vertices incident to u. The stabilizer of u in G induces on Ξ the action of $U_4(2)$ or $U_4(2).2$. So the stabilizer of u in H induces on the vertices of Ξ adjacent to x the action of $2^4.A_5$ or $2^4.S_5$. But this contradicts the above described action of H on $\Delta(x)$.

So $K \neq K_1$, and by the paragraph after Lemma 6.10 and Proposition 6.12, we see that K_0/K_1 is the 10-dimensional section of the Golay cocode considered as a module for M_{22}. By Proposition 3.1, $\mathcal{F} \simeq \mathcal{G}^T_{22}$ and G induces on $\mathcal{F}$ the action of Fi_{22} or $Fi_{22}.2$. We claim that the action is faithful. Let K be the kernel of the action. Then K is contained in K_1, since H/K_1 acts faithfully on $\mathcal{F}$ and hence K stabilizes every element of type 2 in $\mathcal{F}$ incident to x. So K is a normal subgroup of G which stabilizes an incident pair of elements of type 1 and 2 in $\mathcal{G}$. The connectivity of $\mathcal{G}$ implies that $K = 1$. Now we see that $G \simeq Fi_{22}$ or $Fi_{22}.2$. It is clear that there is only one flag-transitive $c.C_3(2)$-geometry acted on flag-transitively by Fi_{22} or $Fi_{22}.2$, so we obtain the isomorphism $\mathcal{G} \simeq \mathcal{H}_{22}$.

Now suppose that $\mathcal{P} \simeq \mathcal{G}(3.M_{22})$. We consider a rank 3 geometry $\mathcal{F}$ the elements of type 1 and 2 of which are defined exactly as in the previous case; the elements of type 3 will be defined in the next paragraph.

There is a morphism of $\mathcal{G}(3.M_{22})$ onto $\mathcal{G}(M_{22})$ and the subgeometry $\mathcal{S}$ in $\mathcal{G}(3.M_{22})$ is the full preimage under this morphism of an analogous subgeometry in $\mathcal{G}(M_{22})$. This means that the subgeometries in $\mathcal{G}(3.M_{22})$ that we are interested in are also in a bijection with the blocks of the system $S(3, 6, 22)$. This means in particular that the normal subgroup Z of order 3 in H/K_1 does not act on the collinearity graph Δ of $\mathcal{F}$. We define the elements of type 3 in $\mathcal{F}$ to be the orbits of Z on the set of elements of type 4 in $\mathcal{G}$. Now, by exactly the same arguments as in the previous case, we prove that $\mathcal{F} \simeq \mathcal{G}^T_{22}$ and that G induces on $\mathcal{F}$ the action of Fi_{22} or $Fi_{22}.2$, and the kernel of the action coincides with Z. Hence $G \simeq 3 \cdot Fi_{22}$ or $3 \cdot Fi_{22}.2$ and the extension is non-split since the automorphism group of $\mathcal{G}(3.M_{22})$ is a non-split extension. So we have the isomorphism $\mathcal{G} \simeq 3.\mathcal{H}_{22}$. □

PROPOSITION 6.16. *Let $\mathcal{G}$ be a flag-transitive $c.C_4(2)$-geometry all of the $c.C_2(2)$-residues of which are isomorphic to $\mathcal{G}(U_4(2))$. Suppose that the P-geometry associated with $\mathcal{G}$ is isomorphic to $\mathcal{G}(M_{23})$. Then $G \simeq \mathcal{H}_{23}$.*

PROOF. Similarly to the previous proof, we define $\mathcal{F}$ to be a rank 3 geometry the elements of which are the connected components of Λ and M and all the elements of type 4 in $\mathcal{G}$. The geometry $\mathcal{G}(M_{23})$ possesses a description in terms of the Steiner system $S(4, 7, 24)$. In these terms, the elements of type 3 incident to an element of type 1 correspond to the 3-element subsets of points and the subgeometries $\mathcal{S}$ correspond to the blocks of the system. This means that the diagram of $\mathcal{F}$ coincides with that of $\mathcal{G}^T_{23}$. The stabilizer in M_{23} of a subgeometry $\mathcal{S}$ is isomorphic to $2^4.A_7$ and has no subgroups of index 2. This immediately implies that $K_0 \neq K_1$ and, by Proposition 6.12, K_0/K_1 is the 11-dimensional section of the Golay cocode considered as a module for M_{23}. Now Proposition 3.1 gives $\mathcal{F} \simeq \mathcal{G}^T_{23}$. By the same arguments as in the first part of the proof of Proposition 6.15, G acts faithfully on $\mathcal{F}$. Hence $G \simeq Fi_{23}$ and $G \simeq \mathcal{H}_{23}$. □

Proposition 5.3 now follows from Propositions 6.15 and 6.16, and this completes our consideration of the Fischer group geometries. But in fact, using [13] it is possible completely to characterize the considered class of $c.C_n(2)$-geometries.

Let $\mathcal{G}$ be a $c.C_4(2)$-geometry all of the $c.C_2(2)$-residues of which are isomorphic to $\mathcal{G}(U_4(2))$. Let G be a flag-transitive automorphism group of $\mathcal{G}$ and let Γ be the collinearity graph of $\mathcal{G}$. Let x be a vertex of Γ (that is, an element of type 1 in $\mathcal{G}$). Then $G(x)$ induces on $\Gamma(x)$ the action of $Sp_8(2)$ on the cosets of its subgroup isomorphic to $2^{10}.L_4(2)$. It follows from the pushing up results proved in [13] that the action of $G(x)$ on $\Gamma(x)$ is faithful. This means that the stabilizer in G of an element of type 2 in $\mathcal{G}$ is isomorphic to $2^{11}.L_4(2)$. By properties of the automorphism groups of the flag-transitive P-geometries of rank 4 we obtain that the P-geometry associated with $\mathcal{G}$ must be isomorphic to $\mathcal{G}(M_{23})$. Finally, since there are no flag-transitive P-geometries having $\mathcal{G}(M_{23})$ as a residue we have completed the proof of Theorem B.

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